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ON THE RELATIVE EQUILIBRIUM OF A GYROSTAT SATELLITE IN THE GENERALIZED LIMITED CIRCULAR PROBLEM OF THREE BODIES*

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A set of relative equilibriums of a gyrostat satellite is considered in the case when its center of mass is at one of the rectilinear or triangular libration points. A clear geometric concept of this set of equilibriums is given. Sufficient conditions of stability are obtained and analyzed.

1. Consider the three bodies M_1, M_2 and M of which M_1 and M_2 are material points or bodies with spherical distribution of mass, and M is a gyrostat.

We denote by m, m_1, m_2 and G, G_1, G_2 the masses and centers of mass of bodies M, M_1, M_2, and assume that $m_1 \gg m, m_2 \gg m$ and points G_1 and G_2 move relative to each other on Keplerian circular orbits of radius a at the orbital angular velocity $\Omega\,(\Omega^2 = f\,(m_1 + m_2)\,a^{-3})$, where f is the gravitational constant.

We introduce the Cartesian coordinate system G_0xyz with origin at the center of mass of bodies M_1 and M_2 whose x-axis passes through points G_1 and G_2, the z-axis is normal to the plane of these points orbits, and the y-axis coincides with the direction of motion of point G_2. The coordinate system rotates about its z-axis at constant angular velocity Ω. The coordinates a_1 and a_2 of points G_1 and G_2 on the x-axis are

$$a_1 = -\frac{a\,(1-p)}{2}, \quad a_2 = \frac{a\,(1+p)}{2}, \quad p = \frac{1-w}{1+w}, \quad w = \frac{m_2}{m_1} \tag{1.1}$$

We take the principal central axes of inertia of the satellite as the axes of the system of coordinates $Gx_1x_2x_3$, and define the gyrostat body position in the system of coordinates $G_0\,xyz$ by coordinates x, y, z of its center of mass and Euler's angles ϑ, ψ, φ or cosines of angles $\alpha_s, \beta_s, \gamma_s$ $(s = 1, 2, 3)$ between axes x_s and x, y, z, respectively, with

$$\pi_1 = \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_3\gamma_3 = 0, \quad \pi_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{1.2}$$

$$\pi_3 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \tag{1.3}$$

The quantities $\alpha_s, \beta_s, \gamma_s$ are expressed in terms of Euler's angles /1/.

Suppose that the gyrostat rotors rotate at angular velocities that are constant relative to the satellite body, and denote by $k_s = \text{const}$ projections of the total gyrostatic moment of the relative motions of rotors on the x_s-axes.

In calculating the force functions U_1 and U_2 of Newtonian attraction of the satellite by points G_1 and G_2 we assume the characteristic dimension l of the satellite to be considerably smaller than the distances $r_i = [(x - a_i)^2 + y^2 + z^2]^{1/2}$ $(i = 1, 2)$ between point G and points G_1 and G_2. Then, neglecting terms of order l^3/r_i^3 and higher, we have for U_1 and U_2 the approximate expressions /1/

$$U_i = \frac{\mu_i m}{r_i} - \frac{3}{2}\frac{\mu_i}{r_i^3}\left(A_1\gamma_{i1}^2 + A_2\gamma_{i2}^2 + A_3\gamma_{i3}^2 - \frac{A_1 + A_2 + A_3}{3}\right)$$

$$\mu_i = fm_i, \quad \gamma_{is} = \frac{1}{r_i}\,[(x - a_i)\,\alpha_s + y\beta_s + z\gamma_s] \quad (i = 1, 2;\ s = 1, 2, 3)$$

where A_s $(s = 1, 2, 3)$ are the principal moments of inertia of the gyrostat, and γ_{is} are the cosines of angles between the x_s axes and radius vector $\mathbf{r}_i$ of point G_i relative to point G.

The altered potential energy of gravitational and inertia forces acting on the satellite are in the system of coordinates $G_0\,xyz$ of the form /2/

$$W = -\frac{1}{2}\sum_{s=1}^{3}(\Omega^2 A_s\gamma_s^2 + 2\Omega k_s\gamma_s) - \frac{1}{2}m\Omega^2(x^2 + y^2) - U_1 - U_2$$

We introduce the dimensionless coordinates

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$$x = ax',\ y = ay',\ z = az',\ x_s = lx_s'\ (s = 1,\ 2,\ 3)$$

and set

$$\varepsilon = l^2/a^2,\ A_s = l^2A_s',\ k_s = \Omega l^2 k_s' \quad (s = 1,\ 2,\ 3) \tag{1.4}$$

The formula for W then assumes the form

$$a^{-2}\Omega^{-2}W = mW_1(x',\ y',\ z') + \varepsilon W_2(x',\ y',\ z',\ \vartheta,\ \psi,\ \varphi) \tag{1.5}$$

$$W_1 = -\frac{1}{2}(x'^2 + y'^2) - \frac{\mu_1'}{r_1'} - \frac{\mu_2'}{r_2'}, \quad r_i = ar_i', \tag{1.6}$$

$$\mu_i = (m_1 + m_2)\mu_i' \quad (i = 1, 2)$$

$$W_2 = \frac{1}{2}\sum_{s=1}^{3}\{[3(h_\alpha\alpha_s^2 + h_\beta\beta_s^2 + h_\gamma\gamma_s^2 + 2h_{\alpha\beta}\alpha_s\beta_s + 2h_{\beta\gamma}\beta_s\gamma_s + 2h_{\gamma\alpha}\gamma_s\alpha_s) - \gamma_s^2] - 2k_s'\gamma_s - hA_s'\}$$

$$h_\alpha = \frac{\mu_1'(x' - a_1')^2}{r_1'^5} + \frac{\mu_2'(x' - a_2')^2}{r_2'^5}, \quad h = \frac{\mu_1'}{r_1'^3} + \frac{\mu_2'}{r_2'^3} \tag{1.7}$$

$$h_\beta = y'^2h_1, \quad h_\gamma = z'^2h_1, \quad h_{\beta\gamma} = y'z'h_1, \quad h_{\gamma\alpha} = z'h_2, \quad h_{\alpha\beta} = y'h_2$$

$$h_1 = \frac{\mu_1'}{r_1'^5} + \frac{\mu_2'}{r_2'^5}, \quad h_2 = \frac{\mu_1'(x' - a_1')}{r_1'^5} + \frac{\mu_2'(x' - a_2')}{r_2'^5}, \quad a_i = aa_i'$$

Equations of the gyrostat relative equilibriums are obtained from the condition of function W stationarity

$$\frac{\partial W_1}{\partial x'} + \frac{\varepsilon}{m}\frac{\partial W_2}{\partial x'} = 0 \quad (x'y'z'), \qquad \frac{\partial W_2}{\partial \vartheta} = \frac{\partial W_2}{\partial \psi} = \frac{\partial W_2}{\partial \varphi} = 0 \tag{1.8}$$

We seek a solution of Eqs.(1.8) of the form

$$x' = x_{(0)}' + \varepsilon x_{(1)}' + o(\varepsilon)\ (x'y'z') \tag{1.9}$$

$$\vartheta = \vartheta_0 + \varepsilon\vartheta_1 + o(\varepsilon)\ (\vartheta\psi\varphi)$$

For the determination of quantities $x_{(0)}',\ y_{(0)}',\ z_{(0)}'$ we have the equations

$$\frac{\partial W_1}{\partial x'} = \frac{\partial W_1}{\partial y'} = \frac{\partial W_1}{\partial z'} = 0$$

which are the same as the equations of relative equilibrium of the limited problem of three bodies in which M is taken as a material point /1/. Hence three rectilinear and two triangular, respectively, L_1, L_2, L_3 and L_4, L_5 libration points are solutions of these equations.

The quantities $\vartheta_0,\ \psi_0,\ \varphi_0$ are determined using the second group of Eqs.(1.8) in which one of the libration points is to be substituted for x', y', z'.

To determine $x_{(1)}',\ y_{(1)}',\ z_{(1)}',\ \vartheta_1,\ \psi_1,\ \varphi_1$ we differentiate Eqs.(1.8) with respect to ε on the assumption that the variables are functions of ε, and then set $\varepsilon = 0,\ x' = x'_{(0)},\ y' = y_{(0)}',\ z' = z_{(0)}',\ \vartheta = \vartheta_0,\ \psi = \psi_0,\ \varphi = \varphi_0$. As the result we obtain the equations

$$\left(\frac{\partial^2 W_1}{\partial x'^2}\right)_0 x_{(1)}' + \left(\frac{\partial^2 W_1}{\partial y'\,\partial x'}\right)_0 y_{(1)}' + \left(\frac{\partial^2 W_1}{\partial z'\,\partial x'}\right)_0 z_{(1)}' = -\frac{1}{m}\left(\frac{\partial W_2}{\partial x'}\right)_0 (x'y'z') \tag{1.10}$$

$$\left(\frac{\partial^2 W_2}{\partial \vartheta^2}\right)_0 \vartheta_1 + \left(\frac{\partial^2 W_2}{\partial\vartheta\,\partial\psi}\right)_0 \psi_1 + \left(\frac{\partial^2 W_2}{\partial\vartheta\,\partial\varphi}\right)_0 \varphi_1 = -\left(\frac{\partial^2 W_2}{\partial x'\,\partial\vartheta}\right)_0 x_{(1)}' - \left(\frac{\partial^2 W_2}{\partial y'\,\partial\vartheta}\right)_0 y_{(1)}' - \left(\frac{\partial^2 W_2}{\partial z'\,\partial\vartheta}\right) z_{(1)}' \quad \begin{pmatrix} x'y'z' \\ \vartheta\psi\varphi \end{pmatrix}$$

2. Let $x_{(0)}',\ y_{(0)}' = 0,\ z_{(0)}' = 0$ be the coordinates of one of the rectilinear libration points. To determine the gyrostat body orientation in its relative equilibrium we use, instead of the group of Eqs.(1.8), the respective equivalent equations of the directional cosines $\alpha_s, \gamma_s\ (s = 1, 2, 3)$ of axes x and z. We obtain these equation from the condition of function W_2 stationarity in which the coordinates of one of the rectilinear libration points are to be substituted for x', y', z'. We obtain

$$W_2^\circ = \frac{1}{2}\sum_{s=1}^{3}(3h_\alpha^\circ A_s'\alpha_s^2 - A_s'\gamma_s^2 - 2k_s'\gamma_s) \tag{2.1}$$

$$h_\alpha^\circ = \frac{\mu_1'(x_{(0)}' - a_1')^2}{r_1'^5} + \frac{\mu_2'(x_{(0)}' - a_2')^2}{r_2'^5}$$

Since α_s, γ_s are related by the equalities (1.2) and (1.3), we substitute for W_2° the function

$$W_{2*}^{\circ}=W_2^{\circ}+3\lambda h_\alpha^{\circ}\pi_1+\frac{1}{2}\nu\pi_2-\frac{3}{2}\sigma h_\alpha^{\circ}\pi_3$$

where λ, ν, σ are undetermined Lagrange multipliers.

The equations of relative equilibrium are of the form

$$\partial W_{2*}^{\circ}/\partial\alpha_s = 3h_\alpha^{\circ}[(A_s'-\sigma)\alpha_s+\lambda\gamma_s]=0 \; (s=1,2,3) \tag{2.2}$$

$$\partial W_{2*}^{\circ}/\partial\gamma_s = 3\lambda h_\alpha^{\circ}\alpha_s+(\nu-A_s')\gamma_s-k_s'=0 \; (s=1,2,3) \tag{2.3}$$

To investigate Eqs.(1.2), (1.3), (2.2), and (2.3) we use the method developed in /3/ for the problem of the gyrostat satellite steady motions in the field of a single attracting center.

We fix $\nu=\nu_0$ and $\alpha_s=\alpha_{s0}$ $(s=1,2,3)$, to satisfy relation (1.3) and solve Eqs.(1.2), (2.2), and (2.3) for $\sigma, \lambda, \gamma_s, k_s'$. We obtain

$$\sigma=\sigma_0=\sum_{s=1}^{3}A_s'\alpha_{s0}^2 \tag{2.4}$$

$$\lambda=\lambda_0=\pm\Big[\sum_{s=1}^{3}A_s'^2\alpha_{s0}^2-\Big(\sum_{s=1}^{3}A_s'\alpha_{s0}^2\Big)^2\Big]^{1/2} \tag{2.5}$$

$$\gamma_i=\gamma_{i0}=\lambda_0^{-1}\Big(\sum_{s=1}^{3}A_s'\alpha_{s0}^2-A_i'\Big)\alpha_{i0} \quad (i=1,2,3) \tag{2.6}$$

$$k_i'=k_{i0}'=3\lambda_0h_\alpha^{\circ}\alpha_{i0}+(\nu_0-A_i')\gamma_{i0} \quad (i=1,2,3) \tag{2.7}$$

for $\lambda_0\neq0$.

When $\lambda_0=0$, it is possible to take for γ_{i0} any value that satisfies Eqs.(1.2), and determine k_{i0}' using (2.3)

$$k_i'=k_{i0}'=(\nu_0-A_i')\gamma_{i0} \; (i=1,2,3) \tag{2.8}$$

It follows from (2.6) and (2.5) that the gyrostat body orientation in its relative equilibrium position is independent of parameter ν which affects only the choice of gyrostatic moments k_{i0}' and the stability of equilibrium.

The expression for λ_0^2 can be taken in the form /3/

$$\lambda_0^2=\sum_{(123)}(A_2'-A_3')^2\alpha_{20}^2\alpha_{30}^2\geqslant0 \tag{2.9}$$

whose right-hand side vanishes only if in the equilibrium position one of the principal central axes of inertia of the gyrostat is collinear with the x-axis. Hence Eqs.(1.2), (1.3), (2.2), and (2.3) are solvable for $\sigma, \lambda, \gamma_s, k_s'$ with any values of ν_0 and α_{s0} linked by equality (1.3).

Thus the gyrostat satellite in its equilibrium position can be directed toward any of the bodies M_1 and M_2 along any arbitrarily chosen in it direction. If the principal central axis of inertia of the gyrostat is not collinear with axis x $(\lambda_0\neq0)$, then to each such direction correspond two dynamically equivalent equilibrium positions depending on the different signs of λ_0, and differing by a 180° turn about an axis collinear with the x-axis. The quantities k_{i0}' differ for these two equilibrium positions by their signs. When the principal central axis of inertia of the gyrostat is collinear with the axis x $(\lambda_0=0)$, any equilibrium position relative to the turn about that axis is possible.

Consider the following two one-parameter sets of relative equilibriums:

$$\begin{gathered}\alpha_{10}=1,\quad \alpha_{20}=0,\quad \alpha_{30}=0,\quad \gamma_{10}=0,\quad \gamma_{20}=\sin\vartheta,\quad \gamma_{30}=\cos\vartheta\\ k_{10}'=0,\quad k_{20}'=(\nu_0-A_2')\sin\vartheta,\quad k_{30}'=(\nu_0-A_3')\cos\vartheta\end{gathered} \tag{2.10}$$

$$\begin{gathered}\alpha_{10}=0,\quad \alpha_{20}=\cos\vartheta,\quad \alpha_{30}=-\sin\vartheta,\quad \gamma_{10}=0,\quad \gamma_{20}=\sin\vartheta,\quad \gamma_{30}=\cos\vartheta\\ k_{10}'=0,\quad k_{20}'=[\nu_0-A_2'+3(A_2'-A_3')h_\alpha^{\circ}\cos^2\vartheta]\sin\vartheta,\\ k_3'=[\nu_0-A_3'+3(A_3'-A_2')h_\alpha^{\circ}\sin^2\vartheta]\cos\vartheta\end{gathered} \tag{2.11}$$

of which the first was studied by Rumiantsev /2/.

In the equilibrium position (2.10) the x_1-axis is collinear with axis x, and the axes x_2 and x_3 lie in a plane parallel to yz, with the x_3-axis at angle ϑ to the z-axis, and the rotor axis is orthogonal to axes x_1 and x.

For solving (2.11), the x_1-axis is collinear with the y-axis, and axes x_2 and x_3 lie in a plane parallel to xz, with the x_3-axis at angle ϑ to the z-axis, and the rotor axis orthogonal to axes x_1 and y. The sets of relative equilibrium positions of form (2.10) and (2.11) correspond to points $\alpha_{i0} = 0$ $(i = 1, 2, 3)$ of the unit sphere great circles (1.3). If the ellipsoid of inertia of the gyrostat is symmetric $(A_1 = A_2)$, the sets (2.10) and (2.11) represent all possible relative equilibrium positions of the gyrostat.

We shall now construct exact solutions (for the problem formulated in Sect.1) of equations (1.8) of the gyrostat relative equilibrium position of form (1.9). The first terms of expansions (1.9) were investigated above in the zero approximation.

Consider the first group of Eqs.(1.10) which will be used for a more precise definition of the position of the gyrostat center of mass in relative equilibrium. For rectilinear libration points we have

$$\left(\frac{\partial^2 W_1}{\partial x'^2}\right)_0 = -1 - 2h^\circ, \quad \left(\frac{\partial^2 W_1}{\partial y'^2}\right)_0 = -1 + h^\circ, \quad \left(\frac{\partial^2 W_1}{\partial z'^2}\right)_0 = h^\circ \tag{2.12}$$

$$\left(\frac{\partial W_2}{\partial x'}\right)_0 = \frac{1}{2}\left(\frac{\partial h}{\partial x'}\right)_0 \sum_{s=1}^{3} (3\alpha_s^2 - 1) A_s', \quad \left(\frac{\partial W_2}{\partial y'}\right)_0 = 0$$

$$\left(\frac{\partial W_2}{\partial z'}\right)_0 = \lambda_0 \left(\frac{\partial h}{\partial x'}\right)_0$$

$$h^\circ = \frac{\mu_1'}{|x_{(0)}' - a_1'|^3} + \frac{\mu_2'}{|x_{(0)}' - a_2'|^3} > 0$$

$$\left(\frac{\partial h}{\partial x'}\right)_0 = -3\left[\frac{\mu_1'(x_{(0)}' - a_1')}{|x_{(0)}' - a_1'|^5} + \frac{\mu_2'(x_{(0)}' - a_2')}{|x_{(0)}' - a_2'|^5}\right]$$

while the second mixed derivatives of W_1 with respect to x', y', z' vanish.

Substituting expressions (2.12) into (1.10) we obtain

$$x_{(1)}' = \frac{1}{2m(1 + h^\circ)}\left(\frac{\partial h}{\partial x'}\right)_0 \sum_{s=1}^{3} (3\alpha_s^2 - 1) A_s' \tag{2.13}$$

$$y_{(1)}' = 0, \quad z_{(1)}' = -\frac{\lambda_0}{mh^\circ}\left(\frac{\partial h}{\partial x'}\right)_0$$

From this we conclude that when $m_1 \neq m_2$ the gyrostat center of mass in the relative equilibrium lies in the xz-plane at a distance from the libration point of the order of smallness of the ratio l^2/a^2. The center of mass lies on the x-axis only when the equilibrium positions at which the principal central axis of inertia is collinear with the z-axis. At all remaining equilibrium positions the center of mass is displaced also in the direction normal to the plane of its orbit by a quantity of the order of smallness of l^2/a^2.

Similar result was obtained in /3/ in the problem of steady motions of a gyrostat satellite in the field of a single attracting center.

The gyrostat center of mass lies at a libration point in the case of relative equilibriums for which the conditions

$$\lambda_0 = 0, \quad 3\,(A_1\alpha_{10}^2 + A_2\alpha_{20}^2 + A_3\alpha_{30}^2) = A_1 + A_2 + A_3$$

are simultaneously satisfied. This occurs only when solutions of (2.10) are obtained under the supplementary condition: $2A_1 = A_2 + A_3$.

If $m_1 = m_2$, all of the above conclusions remain valid for the libration points L_1 and L_3, and for point L_2 the gyrostat center of mass coincides with that point in the relative equilibrium, since then $(\partial h/\partial x')_0 = 0$.

It follows from the second group of Eqs.(1.10) that in the relative equilibrium of the gyrostat the deviation of quantities α_s, γ_s, k_s' $(s = 1, 2, 3)$ from their values obtained by formulas (2.5) — (2.8) is of the order of smallness of l^2/a^2.

3. Let us investigate the stability of the gyrostat relative equilibriums (2.6) on the assumption that its center of mass moves along an unperturbed circular orbit at angular velocity Ω. For this it is necessary to apply to the center of mass control forces /2/ which would compensate the perturbations due to the gyrostat motion about the center of mass. Then the equations of motion of the gyrostat admit the generalized energy integral $T_r + W = \text{const}$, where T_r is the gyrostat kinetic energy in its motion defined by the system of coordinates $G_0\,xyz$.

By virtue of the Lagrange theorem the satellite relative equilibrium is stable with respect to the quantities $\vartheta, \psi, \varphi, \vartheta^\cdot, \psi^\cdot, \varphi^\cdot$, if function W_2 has then an isolated minimum. Sufficient conditions of stability are obtained from the stipulations of the positive definiteness of the second variation $\delta^2 W_2$ in the equilibrium position neighborhood.

In the case of perturbed motion we set

$$\alpha_s = \alpha_{s0} + \xi_s,\ \gamma_s = \gamma_{s0} + \eta_s\ (s = 1,\ 2,\ 3)$$

and obtain

$$\delta^2 W_{2*} = \sum_{s=1}^{3} [3h_\alpha{}^\circ (A_s' - \sigma_0)\xi_s{}^2 + 6\lambda_0 h_\alpha{}^\circ \xi_s \eta_s + (\nu_0 - A_s')\eta_s{}^2] \tag{3.1}$$

where variables ξ_s, η_s are linked by the relations

$$\delta\pi_1 = \sum_{s=1}^{3} (\gamma_{s0}\xi_s + \alpha_{s0}\eta_s) = 0, \quad \delta\pi_2 = 2\sum_{s=1}^{3} \gamma_{s0}\eta_s = 0, \quad \delta\pi_3 = 2\sum_{s=1}^{3} \alpha_{s0}\xi_s = 0 \tag{3.2}$$

Conditions of positive definiteness of the quadratic form (3.1) on the linear manifold (3.2) reduce to the form /4/

$$g_0 > 0,\ 2g_0\nu_0 + g_1 > 0,\ \Delta = g_0\nu_0{}^2 + g_1\nu_0 + g_2 > 0 \tag{3.3}$$

$$g_0 = \sum_{s=1}^{3} (A_s' - \sigma_0)\beta_{s0}{}^2 = \lambda_0^{-2}(\sigma_0 - A_1')(\sigma_0 - A_2')(\sigma_0 - A_3')$$

$$g_1 = -g_0 \sum_{(123)} (A_2' + A_3')\gamma_{10}{}^2 + 3h_\alpha{}^\circ \sum_{(123)} (A_2' - \sigma_0)(A_3' - \sigma_0)\alpha_{10}{}^2 - 3h_\alpha{}^\circ\lambda_0{}^2$$

$$g_2 = g_0 \sum_{(123)} A_2'A_3'\gamma_{10}{}^2 - 3h_\alpha{}^\circ \sum_{s=1}^{3} A_s'\beta_{s0}{}^2 \sum_{(123)} (A_2' - \sigma_0)(A_3' - \sigma_0)\alpha_{10}{}^2 + 3h_\alpha{}^\circ\lambda_0{}^2 \Big[\sum_{s=1}^{3} A_s'\alpha_{s0}{}^2 - 3h_\alpha{}^\circ \sum_{s=1}^{3} (A_s' - \sigma_0)\gamma_{s0}{}^2\Big]$$

$$\beta_{10} = \gamma_{20}\alpha_{30} - \gamma_{30}\alpha_{20} = \lambda_0^{-1}(A_2' - A_3')\alpha_{20}\alpha_{30} \quad (123)$$

which can be represented in the form /3/

$$g_0 > 0, \quad \nu_0 > \nu_2 \quad \left(2g_0\nu_2 = -g_1 + \sqrt{g_1{}^2 - 4g_0 g_2}\right) \tag{3.4}$$

where ν_2 is the greatest of roots ν_1, ν_2 of equation $\Delta = 0$. The discriminant of this equation is nonnegative, since otherwise we would have $\Delta \neq 0$ with $g_0 \neq 0$ and any ν_0, and the sign of Δ would be the same as that of g_0. Since by virtue of the second of conditions (3.3) the degree of instability would then be different for $\nu_0 = \pm N$, where N is a fairly large positive number, namely, it would be 0 and 2 for $g_0 > 0$ and 1 and 3 for $g_0 < 0$, which contradicts the theory of bifurcations /5/.

Conditions (3.4) coincide with conditions obtained in /3/ for the problem of stability of the gyrostat satellite relative equilibriums in a field of a single attracting center. The analysis in /3/ is entirely applicable to the problem considered here.

Let us indicate the sufficient conditions of stability for the relative equilibriums (2.10) and (2.11).

For the solution of (2.10) these conditions are of the form

$$g_0 = A_2' \cos\vartheta + A_3' \sin\vartheta - A_1' > 0,\ \nu_0 > \nu_2 = A_2' \cos^2\vartheta + A_3' \sin^2\vartheta \tag{3.5}$$

and are equivalent to conditions indicated by Rumiantsev in /2/.

To solve (2.11) the trinomial Δ is factorized

$$\Delta = [(A_1' - A_2'\alpha_{20}{}^2 - A_3'\alpha_{30}{}^2)(\nu_0 - A_1') + 3h_\alpha{}^\circ (A_3' - A_2')\alpha_{20}{}^2\alpha_{30}{}^2] \times [\nu_0 - A_2'\alpha_{20}{}^2 - A_3'\alpha_{30}{}^2 + 3h_\alpha{}^\circ (A_3' - A_2')(\alpha_{20}{}^2 - \alpha_{30}{}^2)]$$

and conditions (3.4) assume the form

$$g_0 = A_1' - A_2'\alpha_{20}{}^2 - A_3'\alpha_{30}{}^2 > 0,\ \nu_0 > \nu_1,\ \nu_0 > \nu_2 \tag{3.6}$$

$$\nu_1 = A_1' + 3h_\alpha{}^\circ (A_2' - A_3')^2 \alpha_{20}{}^2\alpha_{30}{}^2 (A_1' - A_2'\alpha_{20}{}^2 - A_3'\alpha_{30}{}^2)^{-1}$$

$$\nu_2 = A_2'\alpha_{20}{}^2 + A_3'\alpha_{30}{}^2 + 3h_\alpha{}^\circ (A_2' - A_3')(\alpha_{20}{}^2 - \alpha_{30}{}^2)$$

4. Let now $x_{(0)}' = p/2,\ y_{(0)}' = \pm\sqrt{3}/2,\ z_{(0)}' = 0$ be the coordinates of one of the triangular libration points. We seek the hydrostat relative equilibriums in the form of expansions (1.9) whose first terms determine equilibriums in the zero approximation, when the gyrostat center of mass is at the triangular libration point and orientation of the gyrostat body is obtained from the second group of Eqs.(1.8). Instead of these equations we use the equivalent equations in directional cosines that are obtained using the condition of stationarity of function W_2 in which coordinates of the triangular libration point must be substituted for x', y', z'. We have

$$W_2^\circ = \frac{1}{2}\sum_{s=1}^{3}\left\{\left[\frac{3}{4}(\alpha_s^2 + 2\sqrt{3}p\alpha_s\beta_s \operatorname{sign} y'_{(0)} - \gamma_s^2\right] A_s' - 2k_s'\gamma_s\right\} \qquad (4.1)$$

We reduce expression (4.1) to a special form by turning unit vectors α and β of axes x and y about the z-axis by angle θ

$$\alpha' = \alpha\cos\theta + \beta\sin\theta, \quad \beta' = -\alpha\sin\theta + \beta\cos\theta$$

where θ is the angle between the unit vectors α' and α. We denote projections of vectors α' and β' on axes x_s by α_s', β_s' $(s = 1, 2, 3)$. Then the expression

$$D_s = d_s^2 + 2\sqrt{3}p\alpha_s\beta_s \operatorname{sign} y'_{(0)} + 3\beta_s^2 \qquad (4.2)$$

in the new variables assumes the form

$$D_s = (1 + 2\sin^2\theta + \sqrt{3}p\sin 2\theta \operatorname{sign} y'_{(0)})\,\alpha_s'^2 + (1 + 2\cos^2\theta - \sqrt{3}p\sin 2\theta \operatorname{sign} y'_{(0)})\beta_s'^2 +$$
$$2(\sin 2\theta + \sqrt{3}p\cos 2\theta \operatorname{sign} y_0')\,\alpha_s'\beta_s'$$

We determine angle θ using the equation

$$\operatorname{tg} 2\theta = -\sqrt{3}p \operatorname{sign} y'_{(0)} \qquad (4.3)$$

As the ratio $w = m_2/m_1$ is varied from 0 to ∞, parameter p varies from $+1$ to -1, and angle θ from $-\pi/6$ to $\pi/6$ when $y'_{(0)} > 0$ and from $\pi/6$ to $-\pi/6$ when $y'_{(0)} < 0$.

Let C be the intersection point of the x-axis with the straight line passing through the triangular libration point and directed as the unit vector β'. For the coordinate $x' = x_C'$ of point C we have

$$x_C' = \frac{1}{2p}(1 + p^2 - \sqrt{1 + 3p^2})$$

As parameter p is varied from $+1$ to -1, the length of segment G_1C, normalized with respect to a and equal

$$x_C' - a_1' = \frac{1}{2p}(1 + p - \sqrt{1 + 3p^2})$$

varies from 0 to 1 and, consequently, point C travels along the x-axis from point G_1 to point G_2.

Formula (4.2) with (4.3) and the relation

$$\alpha_s'^2 + \beta_s'^2 + \gamma_s^2 = 1 \quad (s = 1,\ 2,\ 3)$$

taken into account assumes the form

$$D_s = 2\sqrt{1 + 3p^2}\,\beta_s'^2 - (2 - \sqrt{1 + 3p^2})\,\gamma_s^2 + 2 - \sqrt{1 + 3p^2}$$

Substituting this expression into (4.1) and omitting the unimportant constant term, we obtain for function W_2° the expression

$$\varkappa_1^{-1} W_2^\circ = \frac{1}{2}\sum_{s=1}^{3}(3\varkappa A_s'\beta_s'^2 - A_s'\gamma_s^2 - 2k_s''\gamma_s) \qquad (4.4)$$

where

$$2\varkappa\varkappa_1 = \sqrt{1 + 3p^2}, \quad 4\varkappa_1 = 10 - 3\sqrt{1 + 3p^2}, \quad k_s' = \varkappa_1 k_s'' \qquad (4.5)$$

and β_s', γ_s are related by the equalities

$$\pi_1' = \beta_1'\gamma_1 + \beta_2'\gamma_2 + \beta_3'\gamma_3 = 0, \quad \pi_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \qquad (4.6)$$

$$\pi_3' = \beta_1'^2 + \beta_2'^2 + \beta_3'^2 = 1 \qquad (4.7)$$

Comparison of (4.6) and (4.7) with (2.1), (1.2) and (1.3) shows that problems of the gyrostat relative equilibrium in the cases when its center of mass is at triangular and rectilinear libration points are dynamically equivalent. Hence for the determination of orientation of a gyrostat whose center of mass is at the triangular libration point it is sufficient to substitute in (2.4)— (2.9) β_s', k_s'' and $\varkappa$ for α_s, k_s' and h_α°, respectively. As the result we have

$$\sigma = \sigma_0 = \sum_{s=1}^{3} A_s'\beta_{s0}'^2, \quad \lambda^2 = \lambda_0^2 = \sum_{(123)} (A_2' - A_3')^2 \beta_{20}'^2 \beta_{30}'^2 \tag{4.8}$$

$$\gamma_i = \gamma_{i0} = \lambda_0^{-1}\left(\sum_{s=1}^{3} A_s'\beta_{s0}'^2 - A_i'\right)\beta_{i0}'$$

$$k_i'' = k_{i0}'' = 3\lambda_0 \varkappa \beta_{i0}' + (\nu_0 - A_i')\gamma_{i0} \quad (i = 1, 2, 3)$$

when $\lambda_0 \neq 0$.

If $\lambda_0 = 0$, it is possible to assign to γ_{i0} any values that satisfy equalities (4.6), and determine k_{i0}'' using formulas

$$k_i'' = k_{i0}'' = (\nu_0 - A_i')\gamma_{i0} \quad (i = 1, 2, 3)$$

Thus a gyrostat in position of equilibrium can be oriented toward point C by any arbitrarily selected direction in it. If the principal central axis of inertia of the gyrostat does not coincide with unit vector β' attached to its center of mass $(\lambda_0 \neq 0)$, then to each such direction correspond two dynamically equivalent equilibrium positions of different signs of λ_0 which differ by a 180° turn about the unit vector β'. For these two equilibrium positions the quantities k_{i0}'' are of different signs. If the principal central axis of inertia of the gyrostat coincides with vector β' $(\lambda_0 = 0)$, any equilibrium position in relation to the turn about vector β' is possible.

Consider the following two sets of relative equilibriums:

$$\beta_{10}' = 1, \ \beta_{20}' = 0, \ \beta_{30}' = 0, \ \gamma_{10} = 0, \ \gamma_{20} = \sin\vartheta, \ \gamma_{30} = \cos\vartheta \tag{4.9}$$
$$k_{10}'' = 0, \ k_{20}'' = (\nu_0 - A_2')\sin\vartheta, \ k_{30}'' = (\nu_0 - A_3')\cos\vartheta$$

$$\beta_{10}' = 0, \ \beta_{20}' = \cos\vartheta, \ \beta_{30}' = -\sin\vartheta, \ \gamma_{10} = 0, \ \gamma_{20} = \sin\vartheta, \ \vartheta_{30} = \cos\vartheta \tag{4.10}$$
$$k_{10}'' = 0, \ k_{20}'' = [\nu_0 - A_2' + 3\varkappa(A_2' - A_3')\cos^2\vartheta]\sin\vartheta,$$
$$k_{30}'' = [\nu_0 - A_3' + 3\varkappa(A_3' - A_2')\sin^2\vartheta]\cos\vartheta$$

which are similar to sets (2.10) and (2.11).

In the solution of (4.9) the x_1-axis is collinear with the unit vector β', axes x_2 and x_3 lie in a plane parallel to vectors α' and γ, with the x_3-axis at angle ϑ to the z-axis, and the rotor axis is orthogonal to the x_1-axis and the unit vector β'.

In the equilibrium position (4.10) the x_1-axis is collinear with the unit vector α' and axes x_2 and x_3 lie in a plane parallel to vectors β' and γ, with the x_3-axis at angle ϑ to the z-axis, and the rotor axis is orthogonal to the x_1-axis and vector α'.

The sets of solutions of form (4.9) and (4.10) correspond to points of great circles $\beta_{i0}' = 0$ $(i = 1, 2, 3)$ on the unit sphere (4.7), while for a symmetric gyrostat $(A_1 = A_2)$ represents all relative equilibrium positions.

When $m_1 = m_2$, point C coincides with point G_0, and the unit vectors α', β' coincide with the unit vectors of axes x, y. Solution (4.9) then becomes the solution investigated by Rumiantsev /2/.

Consider the first group of Eqs.(1.10) with the aim of defining more accurately the center of mass position in the relative equilibrium state. In the case of triangular libration points these equations assume in the system of coordinates $Cuvz$ with unit vectors α' and β' of axes u and v, the form

$$\left(\frac{\partial^2 W_1}{\partial u'^2}\right)_0 u_{(1)}' = -\frac{1}{m}\left(\frac{\partial W_2}{\partial u'}\right)_0, \quad \left(\frac{\partial^2 W_1}{\partial v'^2}\right)_0 = -\frac{1}{m}\left(\frac{\partial W_2}{\partial v'^2}\right)_0 \tag{4.11}$$

$$z_{(1)}' = -\frac{3}{2m}\left(p\sum_{s=1}^{3} A_s'\alpha_{s0}\gamma_{s0} + \sqrt{3}\,\mathrm{sign}\, y_{(0)}' \sum_{s=1}^{3} A_s'\beta_{s0}\gamma_{s0}\right)$$

$$u = au', \quad v = av'$$

$$\left(\frac{\partial^2 W_1}{\partial u'^2}\right)_0 = -\frac{3}{4}\left(2 - \sqrt{1 + 3p^2}\right), \quad \left(\frac{\partial^2 W_1}{\partial v'^2}\right)_0 = -\frac{3}{4}\left(2 + \sqrt{1 + 3p^2}\right)$$

which shows that Eqs.(4.1) can be solved for $u_{(1)}', v_{(1)}', z_{(1)}'$ when $p \neq \pm 1$. In the opposite case one of masses m_1 or m_2 is zero, and point C coincides with the center of mass of that of bodies M_1 or M_2 whose mass is nonzero. Then it is possible to consider the variables v and z supplemented by the angular coordinate φ, as cylindrical coordinates of the gyrostat center of mass, with the last two of Eqs.(4.11) yielding corrections for obtaining a more precise definition of the gyrostat center of mass position in its relative equilibrium.

On the basis of (4.1) we conclude that, if for the relative equilibrium the right-hand side of the last equation is nonzero, the plane of the gyrostat center of mass orbit does not pass through points G_1 and G_2.

Sufficient conditions of relative equilibriums (4.8) and (4.9), and (4.10) are obtained from conditions (3.3) and (3.4) also (3.5) and (3.6) by substituting in the latter the quantities β_s', k_s'', $\varkappa$ for α_s, k_s', h_α^*, respectively.

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